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SHORT COMMUNICATION

USE OF THE KIRCHHOFF TRANSFORMATION IN FINITE ELEMENT ANALYSIS

DAN GIVOLI

Department of Aerospace Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel

ABSTRACT

The Kirchhoff transformation, in conjunction with the finite element method, is proposed as a tool in solving non-linear heat conduction problems. A very simple way to obtain the inverse Kirchhoff transformation is shown, using the contour lines of the Kirchhoff variable obtained from a finite element analysis.

KEY WORDS Non-linear heat conduction Kirchhoff transformation Finite element method

INTRODUCTION

Non-linear heat flow problems are often solved by using a finite element spatial discretization. Applying the finite element method directly to the non-linear heat equation results in a non-linear system of ordinary differential equations in time or, if the problem is a steady-state one, in a non-linear system of algebraic equations. In certain cases one may adopt an alternative procedure, consisting of the following steps: (a) apply the *Kirchhoff transformation* to the non-linear heat equation, thus turning it into a *linear* differential equation; (b) solve the linear transformed problem using finite elements; (c) calculate the inverse transform of the finite element results, which is the desired temperature distribution.

To fix ideas, we consider the two-dimensional steady-state heat flow in an homogeneous and isotropic medium, with temperature-dependent thermal conductivity. Let $T(\mathbf{x})$ be the unknown absolute temperature, where $\mathbf{x}$ is the position in a finite two-dimensional domain Ω, with boundary $\Gamma = \Gamma_g \cup \Gamma_h$. The governing equation is:

$$\nabla \cdot (k(T)\nabla T) = 0 \quad \text{in } \Omega \tag{1}$$

where $k(T)$ is the given temperature-dependent thermal conductivity. On the boundary Γ we prescribe the boundary conditions:

$$T = g \quad \text{on } \Gamma_g \tag{2}$$

$$k(T)T_n = h \quad \text{on } \Gamma_h \tag{3}$$

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where g and h are given functions, and T_n is the normal derivative of T on Γ_h. Thus, on Γ_g the temperature is prescribed whereas on Γ_h the normal heat flux is specified. We assume that Γ_g is not empty. The problem (1)–(3) is non-linear due to the T-dependence of k in (1) and (3).

The Kirchhoff transformation of T is defined as:

$$\mathscr{K}[T] = \int_0^T k(\tau)\,\mathrm{d}\tau \tag{4}$$

We shall use the variable $V(x)$ to denote the Kirchhoff transform of the temperature $T(x)$, i.e. $V = \mathscr{K}[T]$. Now, from (4) there follows:

$$\nabla V \equiv \nabla\mathscr{K}[T] = k(T)\nabla T \tag{5}$$

The relation (5) can be used to turn the non-linear problem (1)–(3) into a linear one. By using (4) and (5), we get from (1)–(3):

$$\nabla^2 V = 0 \quad \text{in } \Omega \tag{6}$$

$$V = \mathscr{K}[g] \quad \text{on } \Gamma_g \tag{7}$$

$$V_n = h \quad \text{on } \Gamma_h \tag{8}$$

Equations (6)–(8) constitute a *linear* problem for the Kirchhoff variable $V(x)$.

The use of the Kirchhoff transformation as shown above is a 'mathematical trick' that has been employed extensively to derive analytic solutions to non-linear heat flow problems[1–5]. It has also been used frequently in the context of the boundary element method[6–10]. However, it has rarely been used in the finite element context[11]. There are two reasons for this. First, the linearization by Kirchhoff transformation is not very general, as it can be applied only under certain limitations. For example, it fails when the medium is inhomogeneous or anisotropic, or when convective or radiative boundary conditions are present. Second, the inverse transformation $T = \mathscr{K}^{-1}[V]$, which is needed as a last step to recover the temperature field, is not always easy to calculate. Therefore, it is usually considered preferable to apply the finite element scheme directly to the original non-linear problem, especially when a commercial finite element code is concerned. On the other hand, the common use of the Kirchhoff transformation in boundary element schemes can be explained by the fact that the latter rely much more severely on the linearity of the problem at hand, being based on integral formulations which involve an analytic fundamental solution.

An explicit expression for the inverse Kirchhoff transformation $\mathscr{K}^{-1}[V]$ can be derived analytically when the conductivity function $k(T)$ is sufficiently simple. Three such expressions are given in an Appendix in Reference 10, including the case where $k(T)$ is linear, which is the case considered in Reference 11. When $k(T)$ is not so simple, no explicit expression for $\mathscr{K}^{-1}[V]$ is available.

The purpose of this note is to provide a very simple procedure to obtain the temperature distribution T from the Kirchhoff variable V. We especially consider finite element schemes. The simplicity and efficiency of the method makes use of the Kirchhoff transformation for solving problems of the form (1)–(3) very attractive. The approach can also be applied to the more complicated time-dependent problem.

TWO THEOREMS ON CONTOUR LINES AND THEIR CONSEQUENCES

First, we prove two almost obvious theorems on the contour lines (or level lines or profiles) of the solution T to (1)–(3).

Theorem 1. Every contour line of T is also a contour line of V and *vice versa.*

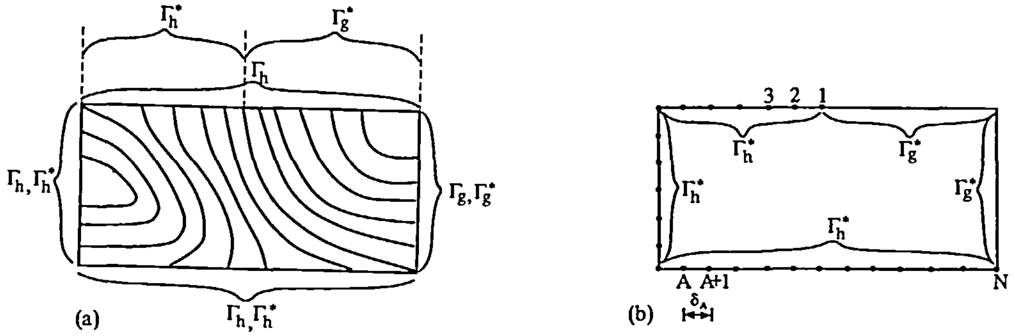

Figure 1 (a) Temperature contour lines, and the boundaries Γ_g, Γ_h, Γ_g^* and Γ_h^*; (b) node and segment numbering on Γ_h^*

Proof. Based on physical reasoning, we assume that the conductivity k is always positive and is a continuous function of T. From (4) we deduce that $V = \mathscr{K}[T]$ is positive. In addition, (4) implies $\mathrm{d}V/\mathrm{d}T = k(T)$ and therefore $\mathrm{d}V/\mathrm{d}T$ is also positive. Hence, V is a continuous and monotonely increasing function of T. By a well known theorem in calculus we conclude that the inverse function $\mathscr{K}^{-1}[V]$ is single-valued. Thus, the mapping between T and V is one-to-one. A contour line of T corresponding to the value T^* is also a contour line of V corresponding to the value $V^* = \mathscr{K}[T^*]$.

Theorem 2. All the contour lines of T intersect the boundary Γ.

Proof. V satisfies Laplace's equation (6). Using the maximum principle for harmonic functions, we deduce that V attains its extremal value on the boundary Γ. From the monotonicity of $V(T)$ indicated above, the temperature T also attains its minimum $T_{\min}$ and maximum $T_{\max}$ on Γ, and there are no local extrema inside Ω. This prevents the possibility of having closed contours inside Ω. Moreover, assuming that the temperature T is a continuous function of position, we conclude that *all* the values of T in the interval $[T_{\min}, T_{\max}]$ are attained on Γ. Therefore, all the contour lines of T intersect Γ.

We make use of these two theorems to devise a procedure for temperature recovery. First, we draw the contour plot corresponding to V, obtained by the finite element analysis. By theorem 1, this is also the contour plot of the temperature T. It remains to associate a temperature value to each contour line. From theorem 2, each contour line intersects either Γ_g or Γ_h. Now, the temperature on Γ_g is *given* by the boundary condition (2). This implies that the T-values of all the contour lines that intersect Γ_g are immediately determined. Furthermore, some of these contour lines have their other end on Γ_h. Thus, the only contour lines whose T-values are not trivially determined are those which have *both* their end points on Γ_h.

In this light, it is appropriate to divide the entire boundary Γ into two new parts. The first part is denoted Γ_g^*, and is defined as the set of all points which are either on Γ_g or on the end point of a contour line starting out from Γ_g. The second part is denoted Γ_h^*, and is defined as the set of all points on Γ_h except those which are on Γ_g^* (see *Figure 1a*). Thus, it remains to find the temperature on Γ_h^*; in turn it immediately determines the temperature values corresponding to all the contour lines in the entire domain Ω. In the next section we describe a simple procedure for calculating T on Γ_h^*.

CALCULATING THE TEMPERATURE ON Γ_h^*

To fix ideas, and without loss of generality, suppose that the boundary Γ_h^* consists of one continuous curve, and that linear (or bilinear) isoparametric finite elements are used near Γ_h^*.

Let the nodes on Γ_h^* be numbered sequentially, i.e. $A = 1, 2, \ldots, N$, where node 1 and node N are the two intersection points of Γ_g^* and Γ_h^* (see *Figure 1b*). We denote T at node A by T_A. Thus, T_1 and T_N are known, but all the other nodal temperatures on Γ_h^* are unknown. We note that V and also its first derivatives are known everywhere, having been found by the finite element analysis. In particular, we know the tangential derivative V_s on the boundary Γ_h^*. It is piecewise constant, and is discontinuous at the nodes on Γ_h^*. We denote the constant value of V_s on the segment between the two consecutive nodes A and $A + 1$ on Γ_h^* by $(V_s)_{A,A+1}$. Thus we have:

$$(V_s)_{A,A+1} = (V_{A+1} - V_A)/\delta_A \tag{9}$$

where δ_A is the length of the segment connecting nodes A and $A + 1$ (see *Figure 1b*).

Now, suppose T_A is given. Then $k(T_A)$ is known. From (5) we have $T_s = V_s/k(T)$, and so approximately:

$$(T_s)_{A,A+1} = \frac{(V_s)_{A,A+1}}{k(T_A)} \tag{10}$$

Also, assuming linear variation of T within the segment $(A, A + 1)$ we can write:

$$(T_s)_{A,A+1} = (T_{A+1} - T_A)/\delta_A \tag{11}$$

We note that the approximations (10) and (11) are somewhat ad-hoc in nature, as opposed to (9) which is consistent with the finite element interpolation. Using (9), (10) and (11), we finally get the recursive formula:

$$T_{A+1} = T_A + \frac{V_{A+1} - V_A}{k(T_A)} \tag{12}$$

Since T_1 is given, we can use (12) recursively to find all the nodal temperatures on Γ_h^*.

It is worth noting that we may repeat the use of (12) starting from the other end of Γ_h^*, namely from node N, and advancing towards node 1. In general, the results would depend on the direction in which we advance. If the results obtained in the two cases are significantly different, this should serve as an indication that the discretization on Γ_h^* must be refined.

Cases where the boundary Γ_h^* consists of more than one continuous curve and where other types of finite elements are used can be treated similarly.

THE MORE GENERAL CASE

The time dependent case is only slightly more involved. In this case (1) is replaced by:

$$\nabla \cdot (k(T)\nabla T) + f = \alpha k(T)\dot{T} \quad \text{in } \Omega \tag{13}$$

Here f is given heat source function, α is the thermal diffusivity, which is assumed to be independent of temperature (otherwise the Kirchhoff transformation does not make (13) linear!), and a dot indicates differentiation with respect to time. An initial condition is supplemented to (13), (2) and (3). In this case theorem 1 still holds, but theorem 2 does not. In other words, if (1) is replaced by the dynamic equation (13) it is possible to have *closed* contour lines Ω. The procedure based on (12) will not be able to determine the T-values corresponding to these closed contours. A direct numerical inversion may be performed to determine those values; if there are only a few closed contours in the contour plot the computational effort involved will be small.

Another possibility is to prepare a table of T *versus* V, for the relevant interval of temperature values. This table is constructed by using the transformation (4) itself, *not its inverse*. After the numerical results for V are available, the temperature associated with each closed contour line is read from the transformation table, using linear interpolation.

The three-dimensional case is technically more complicated. In this case the boundary Γ is a *surface*, and unless the finite element mesh has a high degree of regularity, it is not clear how to automate the procedure for finding T on Γ_h^* using a recursive formula analogous to (12).

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